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SIMPLIFICATIONS IN THE BEHAVIOR OF VISCOELASTIC COMPOSITES WITH GROWING DAMAGE

TECHNICAL REPORT

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SIMPLIFICATIONS IN THE BEHAVIOR OF
VISCOELASTIC COMPOSITES WITH GROWING DAMAGE+

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ABSTRACT

Some analytical results for the mechanical behavior of elastic composite materials and structures with growing damage are summarized and then extended to viscoelastic media. The effect of strain rate only at crack tips is considered first; it is shown that if the crack speed is a strong function of energy release rate, the overall mechanical response is like that for an aging elastic material. Both stable crack growth and unstable crack growth followed by arrest produce this aging-like behavior. Viscoelastic behavior throughout one or more of the phases is then introduced. A simplification is used in which only one relaxation modulus characterizes the viscoelasticity, apart from that at crack tips. Upon replacing the physical displacements in the response for an elastic material by quantities called pseudo displacements, a simple model for viscoelastic composites with growing damage is obtained.

1. Introduction

The problem of developing a realistic mathematical model of the mechanical behavior of viscoelastic composites with growing damage is a difficult one. However, it is believed that considerable simplification may be introduced in the description of both the intrinsic viscoelastic behavior and the damage, while retaining the essential elements needed for a realistic description of deformation and fracture behavior of many composites of engineering interest. The emphasis of this paper is on simplifications which appear to be applicable at least to particle and fiber reinforced polymers when the matrix is soft relative to the particles or fibers. The underlying model for elastic behavior with damage is not restricted in this way.

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We shall not give a general review of the literature in this area; but, instead, contributions of the author on the issue of simplification are emphasized. For a broader view of the subject, the reader is referred to work by Onat and Leckie (1988) and Weitsman (1988), where a tensorial description of damage is addressed. The state variables used in the present paper to characterize global deformation and the damage may be scalars or tensors, but there is no need here to identify them as such. Work on a nonlinear viscoelastic composite-like material, polycrystalline ice, is relevent to the present work. For some studies which address both viscoelasticity and damage growth see Harper (1986, 1989), Karr and Choi (1989), Schapery (1989) and Sjolind (1987).

The present discussion is based in-part on results from Schapery (1990), as summarized in Section 2 where elastic behavior with damage growth is covered. In Sections 3 and 4 we fill in some of the details concerning effects of growing cracks which were only touched upon by Schapery (1990). In addition, some new results are obtained for the case of a body with distributed cracks which become unstable, grow dynamically, and then are arrested. Rate effects at crack tips only are considered in Sections 3 and 4, while simultaneous effects of global and crack-tip viscoelasticity are discussed in Section 5.

2. Elastic Behavior with Damage Growth

We consider an elastic structure or material whose thermodynamic state is a function of independent generalized displacements $q_j (j = 1,2,...J)$ and internal state variables $S_m (m = 1,2,...M)$ as well as temperature or entropy; inelastic behavior arises from changes in the S_m. Generalized forces Q_j are defined in the usual way in that

$$\delta W' = Q_j \delta q_j \quad (j \text{ not summed}) \tag{1}$$

for each virtual displacement δq_j, where $\delta W'$ is the virtual work. Then, from thermodynamics

$$Q_j = \partial W / \partial q_j \tag{2}$$

where W is the Helmholtz free energy (when temperature is used as an independent state variable) or the internal energy (when entropy, instead of temperature, is an independent variable). For brevity, thermal effects will not be considered here, and therefore we shall refer to W simply as the <u>strain energy</u>. The generalized displacements q_j may be, for example, the

uniform strains in a material element and Q_j the conjugate stresses, or q_j and Q_j may be, respectively, the displacements and forces applied to a structure.

The internal state variables serve here to define changes in the structure such as micro- or macro-cracking, and are called <u>structural parameters</u>. Whenever any one $\dot{S}_m \neq 0$, we specify as the evolution law,

$$f_m = \partial W_s / \partial S_m \tag{3}$$

where $W_s = W_s(S_m)$ is a state function of one or more S_m; also, f_m is the <u>thermodynamic force</u>,

$$f_m \equiv -\partial W / \partial S_m \tag{4}$$

The left side of equation (3) is the available force for producing changes in S_m, while the right side is the required force. For any specific set of processes (i.e. histories $q_j(t)$), equation (3) may not be satisfied for all M of the parameters; if it is not, those S_m will be constant. The subscript r or p will henceforth be used in place of m to designate the parameters that change, which are taken to be R in number.

The total work done on the body by Q_j during an <u>actual</u> process (i.e., a process for which parameters change in accordance with equation (3)), starting at some reference state, is denoted by W_T,

$$W_T \equiv \int Q_j dq_j \tag{5}$$

where the summation convention for repeated indices is used. From equations (2)-(5) we find

$$W_T = W + W_s \tag{6}$$

where $W = W_s = 0$ in the reference state. Thus, W_s may be interpreted as that portion of the total work W_T which contributes to changes in the structure.

The second law of thermodynamics provides an inequality as a constraint on the changes in state,

$$\dot{W}_s = T\, S' \geq 0 \tag{7}$$

where T is absolute temperature and S' is the entropy production rate. Even if equation (3) is satisfied for any one S_r, this inequality may not allow it to change. Moreover, instantaneous values of the S_r are such that they minimize the total work when the body passes through stable states; i.e.,

$$\partial W_T / \partial S_r = 0 \tag{8}$$

$$(\partial^2 W_T / \partial S_r \partial S_p) \delta S_r \delta S_p \geq 0 \tag{9}$$

It is observed that equation (3) represents R equations for finding the S_r as functions of q_j. Then, $W_T = W_T(q_j, S_r(q_j), S_q)$ where the S_q are the constant parameters. From equation (5),

$$Q_j = \partial W_T / \partial q_j \tag{10}$$

showing that the body exhibits hyperelastic behavior during the time any particular set of parameters S_r undergoes change. Because the total work is a potential during inelastic processes, the incremental stiffness matrix is symmetric. Conversely, given that the stiffness matrix is symmetric when one or more S_r change, then both equations (3) and (10) follow.

If forces act on crack faces then they have to be included in the set Q_j unless they are associated with frictionless contact; in the latter case, the effect of crack opening and closing may be taken into account through the form of the strain energy function. Coulomb friction, if significant, cannot be accounted for through a work potential, and therefore the stiffness matrix is not necessarily symmetric during processes involving crack face sliding. If, however, one can use a potential to characterize the relationship between crack-face forces and relative displacements between crack faces, equation (10) may be extended to this case by including this potential (which may depend on additional structural parameters) in W_T; such a simplification is applicable with surface free-energy effects (Schapery, 1990) and was proposed by Schapery (1989) to account for crack-face friction in ice under compression.

3. Rate Effects at the Crack Tip

Here we provide the background to Section 4; in that section it is shown that a familiar equation for crack speed, discussed next, leads to equation (3) as an approximation.

Suppose that the local crack tip speed $\dot{a}$ (at an arbitrary point on the crack edge) obeys a power law in local energy release rate G,

$$\dot{a} = k\, G^q \tag{11}$$

where, for now, we assume q is a positive constant; also, $G \equiv -\partial\hat{W}/\partial A$ where $\hat{W}$ is the strain energy of a body with one or more cracks less the surface free energy, and ∂A is an increment in crack surface area. The surface free energy is usually negligible (which we assume here) so that $\hat{W} = W$. The coefficient k may change with time for various reasons, including transient temperatures,

material aging, and mode-ratio effects; if the latter exist, we assume the mode ratio is constant. It will be helpful to write $k = c_1 k_1$, where c_1 is a positive dimensionless function of time and k_1 is a positive constant which has dimensions appropriate for the units used in equation (11). Now, integrate equation (11) with respect to time and then take its q^{th} root,

$$\Delta a^{1/q} = L_G k_1^{\,1/q} \tag{12}$$

where $\Delta a \equiv a - a_o$, a_o is the crack size at t=0, and

$$L_G \equiv [\int_0^{\xi} G^q d\xi']^{1/q} \tag{13}$$

is the so-called Lebesgue norm of G; also,

$$\xi \equiv \int_0^t c_1(t')dt' \tag{14}$$

is <u>reduced time</u>.

If $q = \infty$, then $L_G = G_\ell$, where G_ℓ is the largest value of G up to and including the current time (Reddy and Rasmussen, 1982). For many materials $0 << q < \infty$, which leads to an approximation for L_G which is practically as simple as for $q = \infty$. Consider first the case of power-law reduced-time dependence $G \sim \xi^p$ where $p \geq 0$. Then,

$$L_G = k_2 \xi^{1/q} G \tag{15}$$

in which

$$k_2 \equiv (pq + 1)^{-1/q} \tag{16}$$

The accuracy of equation (15) was studied by Schapery (1982) (using pseudo strain rather than energy release rate as the argument of the Lebesgue norm) for cases in which the argument was a nondecreasing function of ξ, not necessarily a power law. In terms of G used here, let us define p as a logarithmic derivative,

$$p \equiv \frac{d \log G}{d \log \xi} \tag{17}$$

where $p \geq 0$. With $q \geq 4$, good agreement was reported between equation (15) and the value of L_G found by numerical integration for a variety of histories with $p = p(\xi)$. If q is large or p is not strongly time dependent, then k_2 may be taken as a constant, which we shall do here; however, even if p is negative or k_2 is not constant, approximations like that in equation (15) can be developed, and they are useful in view of numerical integration difficulties encounted when $q >> 1$ (Schapery, 1982).

Use of equation (15) in (12) yields

$$G = (\Delta a/\xi)^{1/q}/k_2k_1^{1/q} \tag{18a}$$

which replaces equation (11) as the means for predicting crack growth. The left side is the available work/area for crack growth; therefore the right side may be interpreted as the required work. Equation (18a) is like the crack growth equation for a brittle elastic material, where the right side is the critical fracture energy, say G_c. However, in contrast to brittle elastic behavior, G_c here varies with crack growth and time.

That the Lebesgue norm depends primarily on the current value of G when $q>>1$, rather than its entire history of variation, is obviously due to the integrand (G^q) being a strongly increasing function of G and the assumption $dG/d\xi \geq 0$. Thus, even when the power law equation (11) is not applicable we expect to be able to use a crack growth law of the type

$$G = G_c(\Delta a/\xi) \tag{18b}$$

to predict instantaneous values of Δa. One may arrive at equation (18b) directly by starting with

$$\dot{a} = kf(G) \tag{19a}$$

where k is such that $f(1) = 1$. The form of equation (11) results by using the exponent

$$q \equiv \log f/\log G \tag{19b}$$

(Alternatively, one could define q as a logarithmic derivative instead of a ratio.) Then, if $q>>1$ and $\dot{G} \geq 0$ it is anticipated that equation (18a) will be a good approximation, although its accuracy has not yet been studied. As q is now a function of G, one needs to solve for G; this yields equation (18b), in which G_c is not necessarily a power law in $\Delta a/\xi$.

Another generalization of interest is for cyclic loading when the basic growth law is like equation (11) or (19), but $\dot{a}$ and G are replaced by da/dN and the maximum value of G over a cycle, respectively. Obviously, the above results may be used in this case, but a reduced time based on N, rather than t, enters. With a small modification one may also treat in the same way the case where the amplitude of variation of G over a cycle is used in place of the maximum G (Schapery, 1990).

It should be observed that when $G < G_\ell$ and $q>>1$ then L_G, equation (13), is practically constant if the time period for which $G < G_\ell$ is not extremely long. This implies $\dot{a} \simeq 0$ constant and from (15) that

$$L_G = k_2 \xi_\ell^{1/q} G_\ell \tag{20}$$

where ξ_ℓ is the reduced time at which G first drops below the largest value G_ℓ. This behavior is taken into account in the elastic-like model if we assume $\dot{a} = 0$ when $G < G_c$, where $G_c = G_c(\Delta a/\xi_\ell)$. If G later increases to G_ℓ, say at time ξ_L, then G_c again varies as in equation (18); but ξ should be replaced by $\xi-(\xi_L - \xi_\ell)$ for continuity of G_c. In some cases, such as for cyclic loading, it may be necessary to account for contributions to L_G when G is close to G_ℓ by modifying equation (15) (Schapery, 1982).

In arriving at equation (18) it was not necessary to specify explicitly the manner in which G varys with loading or with geometry of the one or more cracks that may exist. However, this variation certainly will affect the time-dependence of crack growth and thus determine the accuracy of equation (18) and whether or not a physically acceptable solution Δa exists. In order to illustrate this point, let us consider two special cases before discussing the connection between equations (3) and (18).

First, observe that if the energy release rate is constant in time, equation (18) is an exact result. This situation exists for some elementary delamination and transverse microcracking problems in laminates when the applied displacement is constant. A second more interesting case is that for which

$$G = aG_1 \tag{21}$$

where G_1 is a function of only the applied loads or displacements; this form may be derived by dimensional analysis for a linear or nonlinear homogeneous body with an isolated, penny-shaped crack of radius a or straight-edged, through-the-thickness crack of length a. Equation (18a) then can be written as

$$(a - a_o)^{1/q}/a = k_2(k_1\xi)^{1/q}G_1 \tag{22}$$

If $\dot{G}_1 \geq 0$ this equation predicts that $\dot{a}$ is a positive function of ξ for $0<\xi<\xi_f$, where

$$\xi_f \equiv a_o(q-1)^{q-1}/k_1(k_2 q a_o G_1)^q \tag{23}$$

Also, $\dot{a} = \infty$ at $\xi = \xi_f$ and there is no solution for $\xi > \xi_f$. The crack size at time ξ_f is

$$a = a_o q/(q-1) \tag{24}$$

(Note that $a \simeq a_o$ if $q>>1$.) Equation (23) is the limiting time for which a

physically meaningful, stable solution is obtained. Therefore ξ_f may be interpreted as the fracture time, unless the crack growth is arrested by interaction with, for example, originally remote particles or fibers.

When equation (21) is used in the original growth law (11), with q>1, we obtain as the exact solution

$$a/a_o = (1-I_1)^{1/(1-q)} \tag{25}$$

where

$$I_1 \equiv (q-1)k_1 a_o^{(q-1)} \int_0^{\xi} G_1^q d\xi' \tag{26}$$

Observe that crack size is a monotone increasing function of time and $a = \dot{a} = \infty$ at the time for which $I_1 = 1$. Denoting this fracture time by ξ_F, we find

$$\xi_F = \{(q-1)k_1 a_o^{(q-1)} G_1^q\}^{-1} \tag{27}$$

if G_1 is constant. Both ξ_f and ξ_F depend on a_o and G_1 in the same way, and they are equal if we take $k_2 = (q-1)/q$; if q>>1, then $k_2 \simeq 1$, as in equation (16). Thus, for this case in which an instability develops, approximate equation (18a) provides essentially the same crack growth behavior as the exact solution.

4. Work Functions with Rate Effects.

Let us now combine the results in Sections 2 and 3. We assume the instantaneous geometry of all cracks in the body may be defined by the structural parameters S_m. For many particulate and fibrous composites, the cracks tend to be at or close to interfaces or, at least, to have orientations and shapes defined more by the microstructural geometry than by the loading. The orientation of a crack relative to that of the loading will of course affect its rate of growth. Elliptical delaminations, transverse cracks (which are rectangular cracks with planes parallel to fibers and normal to ply surfaces) and cracks between hard particles and a soft matrix are of this type. For these cases it is realistic to use a finite and possibly small number of parameters to define the damage state.

Stable Crack Growth: Use the right side of equation (18a) to define a crack work function for the n^{th} crack,

$$W_n \equiv \int_n \Delta a^{1/q} dA / k_2 (k_1 \xi)^{1/q} \tag{28}$$

where the integral is taken over the area of growth of the n^{th} crack, $A - A_o$, which may not be planar or otherwise regular; the various constants such as q

may be different for different cracks. The growth Δa is a local value which is defined along a curve that is normal to the moving crack edge. If $q = \infty$ this equation is independent of the history of the crack geometry, and yields simply $W_n \sim A - A_0$. Since $1 << q < \infty$, the effect of history is weak, and it is therefore appropriate to use an idealization in which Δa is a single-valued function of A. For example, for cracks which can be idealized as planar, through-the-thickness cracks with straight edges, use $a = A/B$ and $a_0 = A_0/B$ where B is thickness; thus

$$W_n = (A-A_0)^{\lambda}/\lambda k_2(k_1 B\xi)^{1/q} \tag{29}$$

where

$$\lambda \equiv (1+q)/q \tag{30}$$

On the other hand if the cracks are more or less elliptical with moderate aspect ratios or circular use $A = \pi a^2$ in equation (28) and find

$$W_n = 2\pi q(a-a_0)^{1/q}\{a(a-a_0)/q + (a^2 - a_0^2)\}/\lambda(1+2q)k_2(k_1\xi)^{1/q} \tag{31}$$

in which the substitution of

$$a = (A/\pi)^{1/2} \tag{32}$$

into this result gives $W_n = W_n(A,\xi)$. The area of each crack is a function of one or more of the parameters S_m, by previous assumption; if, for example, the cracks are elliptical, we may use for each crack two parameters, the major and minor axes. The total crack work function, considering all cracks, is denoted by W_s,

$$W_s \equiv \sum_n W_n \tag{33}$$

so that $W_s = W_s(S_m, \xi)$, as in equation (3), but now with time-dependence. If q is the same for all cracks, then $W_s \sim \xi^{-1/q}$.

A crack work function based on the more general equation (18b) may be easily derived. For additional generality, also use $A = A_1 a^{\alpha}$, where A_1 and α are positive constants which may vary from crack-to-crack. In this case we find for the n^{th} crack,

$$W_n = A_1 \alpha \xi^{\alpha} \int_0^{\rho} (\rho' + \frac{a_0}{\xi})^{\alpha-1} G_c(\rho')d\rho' \tag{34}$$

where $\rho \equiv \Delta a/\xi$ and $\Delta a = (A/A_1)^{1/\alpha} - a_0$. Then equation (33) yields $W_s(S_m, \xi)$.

We may now use equation (18) to arrive at (3). Let δA be the local

increment in area for a process in which one or more crack edges advance an infinitesimal amount. Multiply equation (18b) by δA and sum along the crack edges, which yields $-\delta W = \delta W_s$. We want this work equality to be valid for all changes $\delta A = (\partial A/\partial S_m)\delta S_m$ due to arbitrary changes δS_m. This requirement yields equation (3), where f_m and W_s are given, respectively, by equations (4) and (33). Inasmuch as $W_s = W_s(S_m,\xi)$, the behavior is the same as for an aging elastic material. However, it should be recalled that the aging stops during periods of "load reduction" (cf. equation (20)).

For the special case in which there are only two different q's, say one $q<\infty$ and one $q=\infty$, equations (28) and (33) provide the simple power-law time-dependence,

$$W_s = W_\infty + W_q \xi^{-1/q} \tag{35}$$

where $W_\infty = W_\infty(S_m)$ and $W_q = W_q(S_m)$. Then, the R parameters S_r which change in a process are found from the R equations,

$$-\frac{\partial W}{\partial S_r} = \frac{\partial W_\infty}{\partial S_r} + \frac{\partial W_q}{\partial S_r}\xi^{-1/q} \tag{36}$$

Even if the q_j are constant in time, equation (36) yields time-dependent values of S_r, and thus time-dependent stresses or forces through equation (2) or (10).

The connection between equation (3) and crack growth theory was established by considering the propagation of individual cracks. However, in modeling damage growth associated with microcracking, it is normally practical to use only a small number of averaging structural parameters which serve to approximate the actual effect of damage on overall mechanical behavior. As done previously (Schapery, 1990) let us require the approximate model to exhibit the same overall limited path-independence of work as implied by equation (10). Inasmuch as equation (3) is necessary and sufficient for such limited path-independence, this growth law should be used in the approximate model. Carrying this argument one step further, we would want the time-dependence of W_s to be essentially the same as that for the more complete representation, e.g. equation (35).

Unstable Crack Growth: The special energy release rate in equation (21) leads to unstable crack growth, which is predicted to occur at a time ξ_f, equation (23), that depends on the initial size. We shall consider a situation in which there are many parallel, planar, penny-shaped cracks with a distribution of radii a_o. It is assumed that when each crack becomes

unstable, it rapidly grows in the original plane to a much larger size and then is arrested by some obstacle. By considering the effect of each crack on overall mechanical response after it reaches its final arrested size, we can develop a work potential W_T which obeys the equations of Section 2. The procedure is analogous to that used by Schapery (1990, Appendix B) for brittle elastic behavior. In this earlier work, generalized forces were used instead of displacements as independent variables. While either set could be used, for consistency with work in the previous sections, we shall use displacements.

The work potential is found to be

$$W_T = W_o - gG_1 + W_s \tag{37}$$

where $W_o = W_o(q_j)$ is the strain energy without cracks and $G_1 = G_1(q_j)$ is the function in equation (21). Also, $g=g(S)$ and $W_s = W_s(S,\xi)$, where

$$W_s = -\frac{(q-1)^Q}{k_2 q (k_1 \xi)^{1/q}} \int_S^{S_1} (S')^{-Q} \frac{dg}{dS'} dS' \tag{38}$$

in which $Q \equiv (q-1)/q$ and S_1 is the initial radius of the largest pre-existing crack. The function g is unspecified here, but depends on the details of microstructure and accounts for the effect of randomly or regularly distributed arrested crack sizes.

The stationary work condition, equation (8), is to be satisfied by the proposed W_T. From equations (37) and (38),

$$\frac{\partial W_T}{\partial S} = \{-G_1 + \frac{(q-1)^Q S^{-Q}}{k_2 q (k_1 \xi)^{1/q}}\} \frac{dg}{dS} \tag{39}$$

The quantity in braces vanishes in view of equation (23) if we consider S to be the initial radius of the crack which becomes unstable at the current time; this equation provides $S = S(G_1, \xi)$. The smaller a crack is, the longer the time is for the crack growth to become unstable, so that $\dot{S}<0$. The body is stable if $\partial^2 W_T/\partial S^2 > 0$; from equation (39) one finds that $dg/dS < 0$. This condition on g not only assures stability, but assures that the entropy production inequality, equation (7), will be satisfied when $\dot{S}<0$. If additional crack orientations are introduced, the work potential will depend in a similar way on additional structural parameters.

We conclude that the work potential based on unstable (micro)crack growth and arrest exhibits the same behavior as that based on stable growth. The power law time-dependence of W_s appears in both cases, except its physical

origin is different. For stable growth, it reflects the direct effect of continuous growth, while for unstable growth it arises from the time delay for instability. The two types of growth may co-exist without changing the form of time-dependence.

5. Global Viscoelasticity

One approach to developing a model which includes viscoelastic effects throughout the matrix and fibers, besides that at crack tips, would be to introduce a rate-type evolution law for a portion of the internal state variables, say $S_\beta (\beta = 1,2,\ldots B)$, of Section 2. For example, for linear viscoelastic behavior with damage given one would use (e.g. Schapery, 1964),

$$\dot{S}_\beta = b_{\beta\gamma} f_\gamma \tag{40}$$

where $b_{\beta\gamma}$ is a symmetric matrix, and $f_\gamma = -\partial W/\partial S_\gamma$ in which W is quadratic in q_j and S_γ. The remaining internal state variables would be associated with the damage, and thus obey equation (3); $b_{\beta\gamma}$ may depend on them. The problem with this approach is that the simple crack growth theory in Section 3 and 4 is not in general applicable because there is not a simple correspondence between elastic and viscoelastic fields in the continuum. Unless considerable simplification is introduced in the description of the global viscoelastic behavior of the constituent materials, it does not appear to be possible to develop a practical analytical model. Here, we shall briefly review a manageable approach the author has used to account for linear and nonlinear viscoelasticity of the matrix; it permits the use of a slightly modified form of the crack growth theory of Sections 3 and 4.

Let us give the constitutive equation without damage and then discuss the modification needed in the elasticity theory with damage. With small strains and rotations the constitutive equation for any one of the constituent materials or phases, in terms of stresses σ_{ij} and strains $\varepsilon_{ij} (i,j = 1,2,3)$, is given as

$$\sigma_{ij} = \partial W_p / \partial \varepsilon^R_{ij} \tag{41a}$$

where $W_p = W_p(\varepsilon^R_{ij})$ and

$$\varepsilon^R_{ij} \equiv E_R^{-1} \int_{-\infty}^{t} E(t-\tau,t) \frac{\partial \varepsilon_{ij}}{\partial \tau} d\tau \tag{41b}$$

are so-called pseudo strains. The quantity $E(t-\tau,t)$ is the relaxation modulus, allowing for aging through the second argument, while E_R is a free constant which can be selected to have the units of modulus so that ε^R_{ij} is

dimensionless. As discussed elsewhere (Schapery, 1981) equation (41) contains the special cases (1) linear isotropic viscoelasticity (if the Poisson's ratio is constant), (2) nonlinear elasticity ($E=E_R$) and (3) linear and nonlinear viscous theory. Inasmuch as W_p is like strain energy density, but is a function of pseudo strains, we call it pseudo strain energy density. It is easily shown that a multiphase continuum may be characterized by equations like (41) if each of the phases obeys equation (41) and all have the same relaxation modulus; phase-to-phase differences are reflected in the particular pseudo strain energy density employed. If the deformations in any phase are relatively small, so that it may be assumed rigid, then of course its relaxation modulus is not restricted to be the same as that for the other phases.

A simple correspondence exists between the mechanical state of elastic and viscoelastic bodies, with or without crack growth, when equations like (41) are applicable (Schapery, 1981, 1984). Large deformations may be taken into account by using Piola stresses and deformation gradients (in place of ε_{ij}); however, there is a basic limitation in that the pseudo strain energy density may be significantly affected by large rotations. This correspondence enables us to use all of the theory in Sections 2-4 by simply replacing q_i with generalized pseudo displacements,

$$q_i^R = E_R^{-1}\int_{-\infty}^{t} E(t-\tau,t)\,\frac{dq_i}{d\tau}\,d\tau \tag{42}$$

while retaining the Q_i as generalized forces. The superscript R comes from use of the name "reference elastic solution" for the set of variables (q_i^R, Q_i); they may be interpreted as the displacements and forces in and on an elastic body which is identical to that of the viscoelastic body except for the relaxation modulus. Observe that when the q_i^R are used in Section 4, there will be hereditary effects due to both damage growth and viscoelasticity of the continuum.

6. Conclusions

An approach to modeling the mechanical response of viscoelastic composites with changing structure has been described. Although the rate-type evolution law used for the changing structure is that commonly identified with crack growth, equations (11) and (19a), it is not necessarily limited to crack growth. Indeed, the approach may be used for any evolution law of the form $\dot{S}_m = F_m(f_m)\hat{F}_m(S_m)$ for each m, where F_m is a strongly increasing function of the associated thermodynamic force f_m. This is a special case of the form used by

Rice (1971) in a study of constitutive relations for metal and other solids; he assumed $\dot{S}_m = \dot{S}_m(f_m, S_1, .., S_M)$ for each m.

Experimental verification of the viscoelastic behavior predicted by the simplified theory described herein is presently very limited. However, existing results on particle-filled rubber with constant and varying damage (Schapery, 1982) and on fiber-reinforced plastic with constant damage (Tonda and Schapery, 1987) do support the theory.

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References

Harper, B.D., 1986, "A Uniaxial Nonlinear Viscoelastic Constitutive Relation for Ice," Journal of Energy Resources Technology, Vol. 108, pp. 156-160.

Harper, B.D., 1989, "Some Implications of a Nonlinear Viscoelastic Constitutive Theory Regarding Interrela- tionships Between Creep and Strength Behavior of Ice at Failure," Journal of Offshore Mechanics and Arctic Engineering, Vol. 111, pp. 144-148.

Karr, D.G. and Choi, K., 1989, "A Three-Dimensional Constitutive Damage Model for Polycrystalline Ice,"Mechanics of Materials, Vol. 8, pp. 55-66.

Onat, E.T. and Leckie, F.A., 1988, "Representation of Mechanical Behavior in the Presence of Changing Internal Structure," Journal of Applied Mechanics, Vol. 55, pp. 1-10.

Reddy, J.N. and Rasmussen, M.L., 1982, Advanced Engineering Analysis, John Wiley & Sons, pp. 203-205.

Rice, J.R., 1971, "Inelastic Constitutive Relations for Solids: An Internal-Variable Theory and its Application to Metal Plasticity," Journal of the Mechanics and Physics of Solids, Vol. 19, pp. 433-455.

Schapery, R.A., 1964, "Application of Thermodynamics to Thermomechanical, Fracture and Birefringent Phenomena in Viscoelastic Media," Journal of Applied Physics, Vol. 35, pp. 1451-1465.

Schapery, R.A., 1981, "On Viscoelastic Deformation and Failure Behavior of Composite Materials with Distributed Flaws," 1981 Advances in Aerospace Structures and Materials, AD-Vol. 1, ASME, Edited by S.S. Wang and W.J. Renton, pp. 5-20.

Schapery, R.A., 1982, "Models for Damage Growth and Fracture in Nonlinear Viscoelastic Particulate Composites," Proc. Ninth U.S. National Congress of

Applied Mechanics, Book No. H00228, ASME, pp. 237-245.

Schapery, R.A., 1984, "Correspondence Principles and a Generalized J Integral for Large Deformation and Fracture Analysis of Viscoelastic Media", International Journal of Fracture, Vol. 25, pp. 195-223.

Schapery, R.A., 1989, "Models for the Deformation Behavior of Viscoelastic Media with Distributed Damage and Their Applicability to Ice," Proc. IUTAM/IAHR Symp. on Ice/Structure Interaction, St. John's, Newfoundland, in press.

Schapery, R.A., 1990, "A Theory of Mechanical Behavior of Elastic Media with Growing Damage and Other Changes in Structure," Journal of the Mechanics and Physics of Solids, Vol. 38, pp. 215-253.

Sjolind, S.G., 1987, "A Constitutive Model for Ice as a Damaging Viscoelastic Material," Cold Regions Science and Technology, Vol. 41, pp. 247-262.

Tonda, R.D. and Schapery, R.A., 1987, "A Method for Studying Composites with Changing Damage by Correcting for the Effects of Matrix Viscoelasticity," Damage Mechanics in Composites, AD-Vol. 12, ASME, Edited by A.S.D. Wang and G.K. Haritos, pp. 45-51.

Weitsman, Y., 1988, "A Continuum Damage Model for Viscoelastic Materials," Journal of Applied Mechanics, Vol. 55, pp. 773-780.